# Online Appendix for *General Airport Slot Allocation Problems*

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## A The Relationship Between G-IA and S-IA

Proposition B.1 is implied by Theorem 1 and Theorem 1 of HR. We give a proof that contains terminologies defined in HR but does not rely on Theorem 1. Note that when $Q = 1$, for any $n \in \{1, 2, ...\}$ with $S_* \cap S_n \neq \emptyset$ and each $a \in A$, $C_n^a$ is either 0 or 1.

**Proposition A.1:** When $Q = 1$, for any $n \in \{1, 2, ...\}$ with $S_* \cap S_n \neq \emptyset$ and each $a \in A$, $C_n^a = 1$ if and only if $s = \Pi_t(f) \in S_n$ with $f \in F_a$ satisfies (E) and (L) at some $\Pi_t$ in the S-IA.

**Proof of Proposition A.1:** ($\Longleftarrow$) Suppose $s = \Pi_t(f) \in S_n$ with $f \in F_a$ satisfies (E) and (L) at some $\Pi_t$ in the S-IA. Let $S' \subset S$ be the block that contains $s$. By Corollary 3 of HR, all type 4 slots in $S'$ are removed with flights of $a$. Note that $\Pi_1$ is ordered and feasible. Since the first slot of $S'$ satisfies (E) at $\Pi_1$, none of these flights, including $f$, can feasibly use a slot earlier than $s$. Therefore, $s$ must be occupied by a flight of $a$ at $\Pi^a$, so $|S_n \cap S^a| = 1$.

Proposition 2 implies a flight that obtains a slot in an earlier block at $\Pi_1$ would also obtain a slot in the same block at $\Pi^a$. Corollary 3 in HR implies all flights in $F_{[s_1,e_f)}$ that obtain a slot in $S'$ at $\Pi_1$ belong to $a$, so each flight in $F_{[s_1,e_f)} \setminus F_a$ obtains a slot in an earlier block. This implies $|F^{\Pi^a,S_{\geq n}} \cap F_{[s_1,e_f)} \setminus F_a| = 0$. By Lemma 4(b) of HR, $s$ satisfies (L) at $\Pi_1$. This implies $F_{[e_f,s_n]} \setminus F_a = \emptyset$. Therefore, we have $|F^{\Pi^a,S_{\geq n}} \cap F_{[e_f,s_n]} \setminus F_a| = 0$. To sum up, we have

$$C_n^a = min\{|S_n \cap S^a|, max[|S_n| - |F^{\Pi^a,S_{\geq n}} \cap F_{[s_1,s_n]} \setminus F_a|, 0]\} = min\{1, max[1, 0]\} = 1.$$

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($\Longrightarrow$) Suppose $s = \Pi_T(f) \in S_n$ with $f \in F_a$ fails (E) and (L) at $\Pi_T$ in the S-IA. By Theorem 1 of HR, $s$ could be used by different airlines at different feasible and non-wasteful landing schedules. If $s$ is not in $S^a$ , then

$$C_n^a = min\{0, max[|S_n| - |F^{\Pi^a, S_{\geq n}} \cap F_{[s_1,s_n]} \setminus F_a|, 0]\} = 0.$$

Suppose $s \in S^a$. Now $|S_n \cap S^a| = 1$. Suppose the number of slots that $a$ obtains in $S_{<n}$ at $\Pi_1$ is $x$, and the number of slots that $a$ obtains in $S_{<n}$ at $\Pi^a$ is $y$. Lemma 2 implies that $x \leq y$. By Proposition 1 of HR, the sets of occupied slots at $\Pi_1$ and $\Pi^a$ are the same.

Case 1: If $x < y$, then a flight of another airline that obtains a slot in $S_{<n}$ at $\Pi_1$ obtains a slot in $S_{\geq n}$ at $\Pi^a$.

Case 1: Suppose $x = y$. Lemma 1 implies that at $\Pi^a$, the flights of $a$ obtain the maximum number of slots that they can obtain in $S_{<n}$ at any feasible and non-wasteful landing schedule, and thus flights in $F_{[s_1,s_n]} \setminus F_a$ obtain the minimum number of slots that they can obtain in $S_{<n}$ at any feasible and non-wasteful landing schedule. Now $s$ can be feasibly and non-wastefully used by another airline means there must exist a flight that can use $s$ but obtain a slot in $S_{\geq n}$ at $\Pi^a$.

In both cases, $|F^{\Pi^a, S_{\geq n}} \cap F_{[s_1,s_n]} \setminus F_a| \geq 1$. Therefore, $|S_n| - |F^{\Pi^a, S_{\geq n}} \cap F_{[s_1,s_n]} \setminus F_a| \leq 0$, we have $C_n^a = min\{1, 0\} = 0$. ■

# B A Class of Lottery Mechanisms

In the Online Appendix of HR, they provide a lottery mechanism that uses a *random ordering algorithm* that provides better incentives in some cases. We generalize their lottery mechanism in this section.

An airline can freeze its flights in slots that it owns, and flights might be canceled in the GDP or before the GDP starts.[1] For each $a \in A$, let $k_a$ be the number of frozen flights of $a$, $m_a$ be the number of canceled flights of $a$, and $o_a$ be the number of originally scheduled flights of $a$. $F_a$ is indeed the set of non-canceled and non-frozen flights of $a$. We assume $o_a = k_a + m_a + |F_a|$. Let $n_a = m_a + |F_a|$ and $n = (n_a)_{a \in A}$. We extend an instance in the LP model by including $n$, so $I = (A, F_\dagger, S, \Phi_\dagger, e, R, n)$.

A *(direct) lottery mechanism* selects a schedule lottery $\phi(I)$ for each instance $I$. We define a class of lottery mechanisms: a *general multiple trading cycles mechanism with random ordering process (GMTCR)* is a lottery mechanism that selects a schedule lottery for each $I$

[1] "Airlines will also have the capability to freeze flights they don't want moved up through the submission of an earliest time of arrival" (?).

using a *random ordering process* and the GMTC algorithm:

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| **Random Ordering Process:** Randomly select an ordering $z^{EA}$ from a given distribution over $Z^E \cap Z^A$ and call it $z$. |
| **GMTC Algorithm:** As in the main text. |

**Corollary A.1:** Any GMTCR is ex post feasible, ex post non-wasteful, ex post individually rational, and ex post Pareto efficient.[2]

This corollary is immediate from Proposition 4. We define the *general multiple trading cycles mechanism with random ordering algorithm*, $\phi^{GM}$, to be a lottery mechanism that selects a schedule lottery for each $I$ using the random ordering algorithm and the GMTC algorithm.

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| **Random Ordering Algorithm:** Create $n_a$ copies of $a$ for each $a \in A$. Draw a copy at a time without replacement. Denote the first copy of $a$ by $a(1)$, the second copy of $a$ by $a(2)$, and so on. Denote the resultant ordering by $z^\star$. For each $a$, eliminate each $a(i)$ with $i > \lvert F_a \rvert$ from $z^\star$ and denote the resultant ordering by $z$. |
| **GMTC Algorithm:** As in the main text. |

HR indicate that assigning $n_a$ slots to each $a \in A$ is consistent with the current mechanism. $\phi^{GM}$ only assigns $|F_a|$ slots to each $a \in A$. We suggest a supplementary algorithm for $\phi^{GM}$. Suppose $z^\star$ and $z$ realized in the random ordering algorithm, and $\phi^z(I)$ is the realized landing schedule of $\phi^{GM}$. Let $\Phi^{\phi^z(I)}$ be the induced slot ownership function of $\phi^z(I)$. The following supplementary algorithm generalizes the one in HR and amends $\Phi^{\phi^z(I)}$ by assigning an additional $M_a$ slots to each $a \in A$. Denote the resultant slot ownership function by $\Phi^{\phi^{z^\star}(I)}$.

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| For each $a$, eliminate each $a(i)$ with $i \leq \lvert F_a \rvert$ from $z^\star$ and denote the resultant ordering by $\mathfrak{z}^1$. Let $V_1 = S \setminus S_1$. For $t \in \{1, 2, ...\}$, repeat the following: Find $s$, which is the earliest slot with the lowest index in $V_t \cap S_A$ that satisfies the following requirement: $s \in S_a$ for some $a$ and $a$ has a surrogate in $\mathfrak{z}^t$. Assign the slot to $a$ and remove the last surrogate of $a$ from $\mathfrak{z}^t$. Update $\mathfrak{z}^t$ to $\mathfrak{z}^{t+1}$ and $V^t$ to $V^{t+1}$. Stop if no slot satisfies the above requirement. If there is no remaining surrogate, stop; otherwise, denote the resultant ordering by $\mathfrak{z}^T$ and assign the earliest unassigned slots (start from the lowest indices) to the airlines sequentially according to $\mathfrak{z}^T$. |

# C Extra Examples

**Example C.1:**

[2] A lottery mechanism is *ex post feasible*, *ex post non-wasteful*, *ex post individually rational*, and *ex post Pareto efficient* if for any instance, it only gives positive probabilities to landing schedules that are feasible, non-wasteful, individually rational, and Pareto efficient, respectively.

| $F$ | $f_{a,1}$ | $f_{a,2}$ | $f_{b,1}$ | $f_{b,2}$ |
|---|---|---|---|---|
| $e$ | 1 | 3 | 4 | 1 |
| $R$ | 1 | 2 | 1 | 2 |
| $e'$ | 1 | 4 | 4 | 1 |

| $S$ | $s_1^1$ | $s_2^1$ | $s_3^1$ | $s_4^1$ | $s_5^1$ |
|---|---|---|---|---|---|
| $\varphi^{GZ}(I)$ | $f_{b,2}$ | $f_{a,1}$ | $f_{a,2}$ | $f_{b,1}$ | |
| $\varphi^{GZ}(I')$ | $f_{a,1}$ | $f_{b,2}$ | | $f_{b,1}$ | $f_{a,2}$ |

This example is from HR, and we use it to show $\varphi^{GZ}$ is not strategy-proof.[3] Suppose $\varphi^{GZ}$ is employed and $z = b(1), a(1), b(2), a(2)$. $C_3^a = 1$ and $C_4^b = 1$. In step 1, $s_4^1 \in (S^1 \setminus S^{uc,1} \cap S_4) \cap (S_b \cup S_{-A} \cup S_{\cancel{A}^1})$. Because $C_4^b = 1$, now $b(1)$ represents $f_{b,2}$, $b(2)$ represents $f_{b,1}$, and $s_{f_{b,1}} = s_4^1$. $b(1)$ and $s_1^1$ form a cycle. In step 2, $(a(1), s_2^1)$ is a cycle. In step 3, $(b(2), s_4^1)$ is a cycle. In step 4, $s_3^1 \in (S^4 \setminus S^{uc,2} \cap S_3) \cap (S_a \cup S_{-A} \cup S_{\cancel{A}^4})$. Because $C_3^a = 1$, $a(2)$ represents $f_{a,2}$ and $s_{f_{a,2}} = s_3^1$. $a(2)$ and $s_3^1$ form a cycle.

Denote the instance where $a$ reports $e'_{f_{a_1}} = 1$ and $e'_{f_{a_2}} = 4$ by $I'$. In step 1, $(b(1), s_4^1)$ is a cycle. In step 2, $(a(1), s_1^1)$ is a cycle. In step 3, $(b(2), s_2^1)$ is a cycle. In step 4, $(a(2), s_5^1)$ is a cycle. By misreporting the earliest feasible arrival times of its flights, $a$ obtains slot $s_1^1$ instead of $s_2^1$ for $f_{a,1}$.[4]

**Example C.2:**

| $F$ | $f_{a,1}$ | $f_{a,2}$ | $f_{b,1}$ | $f_{b,2}$ | $f_{c,1}$ |
|---|---|---|---|---|---|
| $e$ | 4 | 1 | 3 | 1 | 5 |
| $R$ | 1 | 2 | 1 | 2 | 1 |

| $S$ | $s_1^1$ | $s_2^1$ | $s_3^1$ | $s_4^1$ | $s_5^1$ | $s_6^1$ |
|---|---|---|---|---|---|---|
| $\varphi^{GZ}(I)$ | $f_{b,2}$ | $f_{a,2}$ | $f_{b,1}$ | $f_{a,1}$ | $f_{c,1}$ | |
| $\varphi^{GZ}(I')$ | $f_{a,2}$ | $f_{b,2}$ | $f_{b,1}$ | $f_{a,1}$ | | $f_{c,1}$ |

This example shows that $\varphi^{GZ}$ is manipulable via slot destruction.[5] Suppose $\varphi^{GZ}$ is employed and $z = b(1), a(1), a(2), b(2), c(1)$. Suppose $S_a = \{s_5^1\}$, $S_b = \{s_6^1\}$, and $S_c = \{s_3^1, s_4^1\}$. $C_4^a = 1$, $C_3^b = 1$, and $C_5^c = 1$. In step 1, $(a(1), s_4^1, c(1), s_5^1)$ is a cycle, while $b(1)$ points to $s_3^1 \in S_c$. In step 2, $c \in \cancel{A}^2$, so $s_3^1 \in (S^2 \setminus S^{uc,1} \cap S_3) \cap (S_b \cup S_{-A} \cup S_{\cancel{A}^2})$. Because $C_3^b = 1$, now $b(1)$ represents $f_{b,2}$, $b(2)$ represents $f_{b,1}$, and $s_{f_{b,1}} = s_3^1$. $b(1)$ and $s_1^1$ form a cycle. In step 3, $(a(2), s_2^1)$ is a cycle. In step 4, $(b(2), s_3^1)$ is a cycle.

Now suppose $a$ destroys $s_5^1 \in S_a$. Denote the instance where $s_5^1$ is destroyed by $I'$. In step 1, $(b(1), s_3^1, c(1), s_6^1)$ is a cycle. In step 2, $c \in \cancel{A}^2$, so $s_4^1 \in (S^2 \setminus S^{uc,1} \cap S_4) \cap (S_a \cup S_{-A} \cup S_{\cancel{A}^2})$. Because $C_4^a = 1$, now $a(1)$ represents $f_{a,2}$, $a(2)$ represents $f_{a,1}$, and $s_{f_{a,1}} = s_4^1$. $a(1)$ and $s_1^1$ form a cycle. In step 3, $(a(2), s_4^1)$ is a cycle. In step 4, $(b(2), s_2^1)$ is a cycle. By destroying $s_5^1$, $a$ obtains slot $s_1^1$ instead of $s_2^1$ for $f_{a,2}$.

[3] The definition is unnecessary here. See HR for more details.
[4] By updating the earliest feasible arrival time of $f_{a,2}$ to $s_3$ later, $a$ might also obtain $s_3^1$.
[5] The definition is unnecessary here. See HR for more details.