Online Appendix for *Dynamic College Admissions Problem*

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August 31, 2021

Dynamic Core

A matching μ is **period-1 dominated** by another matching $\tilde{\mu}$ via a coalition $L \subseteq S \cup U$ if (i) $\forall l \in L$, $i \in \tilde{\mu}^t(l)$ implies $l \in \tilde{\mu}^t(i)$ and $i \in L$, (ii) $\forall s \in L$, $\tilde{\mu}(s) \succsim_s \mu(s)$ and $\forall u \in L$, $\tilde{\mu} \succsim_u \mu$, and (iii) $\exists s \in L$, $\tilde{\mu}(s) \succ_s \mu(s)$ or $\exists u \in L$, $\tilde{\mu} \succ_u \mu$. A matching μ is **period-2 dominated** by another matching $\tilde{\mu} = (\mu^1, \tilde{\mu}^2)$ via a coalition $L \subseteq S \cup U$ if (i) $\forall s \in L$, $\tilde{\mu}^2(s) = u$ implies $s \in \tilde{\mu}^2(u) \setminus \mu^1(u)$ and $u \in L$, (ii) $\forall u \in L$, $s \in \tilde{\mu}^2(u) \setminus \mu^1(u)$ implies $\tilde{\mu}^2(s) = u$ and $s \in L$, (iii) $\forall s \in L$, $\tilde{\mu}(s) \succsim_s \mu(s)$ and $\forall u \in L$, $\tilde{\mu} \succsim_u \mu$, and (iv) $\exists s \in L$, $\tilde{\mu}(s) \succ_s \mu(s)$ or $\exists u \in L$, $\tilde{\mu} \succ_u \mu$. The set of matchings that are not period-t dominated by any other matching is the **dynamic core**. It is easy to see that the dynamic core is inside the Pareto set.[1] Recall that for any acceptable matchings μ and $\tilde{\mu}$, $\mu^2(u) P^*_u \tilde{\mu}^2(u)$ if and only if $\mu \succ_u \tilde{\mu}$, and $\mu^2(u) R^*_u \tilde{\mu}^2(u)$ if and only if $\mu \succsim_u \tilde{\mu}$.

Theorem A.1: Under A1, a matching is in the dynamic core if and only if it is dynamically stable.

Proof of Theorem A.1: ($\implies$) Suppose μ is not individually rational. If $\emptyset \succ_s \mu(s)$ for some $s \in S$, then μ is not in the dynamic core since it is period-1 dominated by a matching $\tilde{\mu}$ with $\tilde{\mu}(s) = \emptyset$ via a coalition $L = \{s\}$. If $\emptyset \succ_u \mu(u)$ for some $u \in U$, then it is period-1 dominated by a matching $\tilde{\mu}$ with $\tilde{\mu}^1(u) = \emptyset$ and $\tilde{\mu}^2(u) = \emptyset$ via a coalition $L = \{u\}$.

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[1] A matching μ is **Pareto dominated** by $\tilde{\mu}$ if (i) $\forall s \in S$, $\tilde{\mu}(s) \succsim_s \mu(s)$ and $\forall u \in U$, $\tilde{\mu} \succsim_u \mu$, and (ii) $\exists s \in S$, $\tilde{\mu}(s) \succ_s \mu(s)$ or $\exists u \in U$, $\tilde{\mu} \succ_u \mu$. A matching μ is **Pareto efficient** if it is not Pareto dominated by any other matching, and we call the set of matchings that are not Pareto dominated by any other matching the **Pareto set**.

Suppose μ is individually rational but blocked by some $s \in S$ and $u \in U$. Note that μ eliminates justified internal envy. Recall that A1 implies $S_u = S$ for $u \in U$. Therefore, μ strongly eliminates justified internal envy. When s and u period-1 block μ, either s is not matched with u at μ, or s is matched with u at μ and takes an empty position in another period (in this situation, the set of students that are matched with u is unchanged). The situation where s is matched with u at μ and takes an occupied position in another period is incompatible with the fact that μ strongly eliminates justified internal envy, which says that there does not exist $s, s' \in \mu^2(u)$ such that sP_us' and $\mu(s') \succ_s \mu(s)$.

Suppose μ is period-1 blocked by $\{s, u\}$ with a type-1 plan. Then it is period-1 dominated by a matching $\tilde{\mu}$ with $\tilde{\mu}(s) = uu$, $\tilde{\mu}^1(u) = s \cup \mu^1(u) \setminus \sigma$, and $\tilde{\mu}^{2\setminus 1}(u) = \mu^{2\setminus 1}(u)$ via a coalition $L = u \cup s \cup \mu^2(u) \setminus \sigma$.[2]

Suppose μ is period-1 blocked by $\{s, u\}$ with a type-2 plan. Then it is period-1 dominated by any matching $\tilde{\mu}$ with $\tilde{\mu}(s) = cu$, $\tilde{\mu}^1(u) = \mu^1(u)$, and $\tilde{\mu}^{2\setminus 1}(u) = s \cup \mu^{2\setminus 1}(u) \setminus \sigma$ via a coalition $L = u \cup s \cup \mu^2(u) \setminus \sigma$.

Suppose μ is period-2 blocked by $\{s, u\}$. Then it is period-2 dominated by a matching $\tilde{\mu}$ with $\tilde{\mu}^2(s) = u$ and $\tilde{\mu}^2(u) \setminus \mu^1(u) = s \cup \mu^{2\setminus 1}(u) \setminus \sigma$ via a coalition $L = u \cup s \cup \mu^{2\setminus 1}(u) \setminus \sigma$.

($\Longleftarrow$) Suppose μ is not in the dynamic core. Then μ is period-1 or period-2 dominated by a matching $\tilde{\mu}$ via some coalition L, and hence some student s prefer $\tilde{\mu}(s)$ to $\mu(s)$ or some university prefer $\tilde{\mu}$ to μ. If μ is not individually rational, then it is not dynamically stable. Suppose μ satisfies individual rationality.

Case 1: Suppose μ is period-2 dominated by some matching $\tilde{\mu} = (\mu^1, \tilde{\mu}^2)$ via some coalition L.

Case 1.1: Suppose $\tilde{\mu}^2(u) \setminus \mu^1(u) P_u^* \mu^{2\setminus 1}(u)$ for some $u \in L$. There must exist $s \in \tilde{\mu}^2(u) \setminus \mu^1(u) \setminus \mu^2(u)$ and $\sigma \subseteq \mu^{2\setminus 1}(u) \setminus \tilde{\mu}^2(u)$ such that $sP_u\sigma$; otherwise, $\sigma R_u^* s$ for each $s \in \tilde{\mu}^2(u) \setminus \mu^1(u) \setminus \mu^2(u)$ implies $\mu^{2\setminus 1}(u) R_u^* \tilde{\mu}^2(u) \setminus \mu^1(u)$ (via responsiveness). By definition, $s \in \tilde{\mu}^2(u) \setminus \mu^1(u)$ implies $s \in L$, so $cu = \tilde{\mu}(s) \succsim_s \mu(s)$ (since A1 is assumed, only cu is possible here). Since preferences are strict and $s \notin \mu^{2\setminus 1}(u)$, $\tilde{\mu}(s) \neq \mu(s)$ and thus $\tilde{\mu}(s) \succ_s \mu(s)$. Hence, $\{s, u\}$ period-1 block μ with a type-2 plan.

Case 1.2: Suppose $\tilde{\mu}(s) \succ_s \mu(s)$ for some $s \in L$ with $\tilde{\mu}^2(s) = u$. This implies $s \in \tilde{\mu}^2(u) \setminus \mu^1(u)$ and $u \in L$, which in turn implies $\tilde{\mu}^2(u) \setminus \mu^1(u) R_u^* \mu^{2\setminus 1}(u)$. $s \in \tilde{\mu}^2(u) \setminus \mu^1(u)$ implies $s \notin \mu^1(u)$. Then $\tilde{\mu}^2(s) = u$ with A1 implies $\tilde{\mu}(s) = cu$. Then $\tilde{\mu}(s) \succ_s \mu(s)$ implies $\mu(s) \neq cu$ and thus $s \notin \mu^{2\setminus 1}(u)$. Therefore, $\tilde{\mu}^2(u) \setminus \mu^1(u) \neq \mu^{2\setminus 1}(u)$. Hence, $\tilde{\mu}^2(u) \setminus \mu^1(u) P_u^* \mu^{2\setminus 1}(u)$. This implies there exits $\sigma \in \mu^{2\setminus 1}(u) \setminus \tilde{\mu}^2(u)$ such that $sP_u\sigma$. Hence, $\{s, u\}$ period-1 block μ with a type-2 plan.

Case 2: Suppose μ is period-1 dominated by some matching $\tilde{\mu}$ via some coalition L.

[2] We use u to denote the singleton $\{u\}$ if no confusion arises.

Case 2.1: Suppose $\tilde{\mu}^2(u)P_u^*\mu^2(u)$ for some $u \in L$. There must exist $s \in \tilde{\mu}^2(u) \setminus \mu^2(u)$ and $\sigma \subseteq \mu^2(u) \setminus \tilde{\mu}^2(u)$ such that $sP_u\sigma$; otherwise, $\sigma R_u^* s$ for each $s \in \tilde{\mu}^2(u) \setminus \mu^2(u)$ implies $\mu^2(u)R_u^*\tilde{\mu}^2(u)$ (via responsiveness). By definition, $s \in \tilde{\mu}^2(u)$ implies $s \in L$, so $\tilde{\mu}(s) \succsim_s \mu(s)$; furthermore, $s \in \tilde{\mu}^2(u) \setminus \mu^2(u)$ implies $\tilde{\mu}(s) \neq \mu(s)$. Therefore, $\tilde{\mu}(s) \succ_s \mu(s)$. Hence, $\{s, u\}$ period-1 block μ with $\tilde{\mu}(s)$.

Case 2.2: Suppose $\tilde{\mu}(s) \succ_s \mu(s)$ for some $s \in L$ with $\tilde{\mu}^2(s) = u$. This implies $u \in L$, so $\tilde{\mu}^2(u)R_u^*\mu^2(u)$.

Case 2.2.1: If $\tilde{\mu}^2(u) \neq \mu^2(u)$, then $\tilde{\mu}^2(u)P_u^*\mu^2(u)$. This implies there is a student $s' \in \tilde{\mu}^2(u) \setminus \mu^2(u)$ (possibly different from s) and $\sigma \subseteq \mu^2(u) \setminus \tilde{\mu}^2(u)$ such that $s'P_u\sigma$. $s' \in \tilde{\mu}^2(u)$ implies $s' \in L$ and thus $\tilde{\mu}(s') \succsim_{s'} \mu(s')$. $s' \notin \mu^2(u)$ implies $\tilde{\mu}(s') \neq \mu(s')$. So $\tilde{\mu}(s') \succ_{s'} \mu(s')$. Then $\{s', u\}$ period-1 block μ with $\tilde{\mu}(s')$.

Case 2.2.2: If $\tilde{\mu}^2(u) = \mu^2(u)$, then s switches between a type-1 plan and a type-2 plan. Since preferences are strict, without loss of generality, suppose $uu \succ_s cu$ (the arguments are the same when $cu \succ_s uu$). Now $\tilde{\mu}(s) = uu$.

Case 2.2.2.1: There is an empty period-1 position in u at μ. Then $\{s, u\}$ period-1 block μ with a type-1 plan.

Case 2.2.2.2: There exists some student s' such that $\mu(s') = uu$ and $\tilde{\mu}(s') = cu$. Because $s' \in L$, $cu = \tilde{\mu}(s') \succsim_{s'} \mu(s') = uu$. Since preferences are strict, $cu \succ_{s'} uu$. Therefore, $\{s, u\}$ period-1 block μ with a type-1 plan if sP_us', and $\{s', u\}$ period-1 block μ with a type-2 plan if $s'P_us$. $\square$